J Glob Optim (2009) 43:171–174
DOI 10.1007/s10898-008-9279-9

A note on a paper "A regularization method for the proximal point algorithm"

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Received: 14 July 2007 / Accepted: 26 January 2008 / Published online: 14 February 2008
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Abstract In this note, a small gap is corrected in the proof of H.K. Xu [Theorem 3.3, A regularization method for the proximal point algorithm, J. Glob. Optim. **36**, 115–125 (2006)], and some strict restriction is removed also.

Keywords Maximal monotone operator · Proximal point algorithm · Resolvent identity · Strong convergence

AMS Subject Classification (2000) 49J40 · 47J20 · 65J15

The paper [1] deals with the strong convergence of a regularization method for Rockafellar type proximal point algorithm. Particularly, H.K Xu [1] proved the following:

Theorem X ([1, Theorem 3.3]) *Let T be a maximal monotone operator of a Hilbert space H with $S = T^{-1}0 \neq \emptyset$. Let J_c^T be the resolvent of T, $J_c^T = (I + cT)^{-1} (\forall c > 0)$ and $\{x^k\}$ be defined by*

$$x^{k+1} = J_{c_k}^T (t_k u + (1 - t_k)x^k + e^k).$$

Suppose that $\{t_k\} \subset (0, 1)$ and $\{c_k\} \subset (0, +\infty)$ satisfy the following conditions: (i) $\lim_{k\to\infty} t_k = 0$, (ii) $\sum_{k=0}^{+\infty} t_k = \infty$, (iii) $\sum_{k=0}^{+\infty} |t_{k+1} - t_k| < \infty$, (iv) *there are constants* $0 < \underline{c} \leq \overline{c}$ *such that* $\underline{c} \leq c_k \leq \overline{c}$ *for all* $k \geq 0$ *and* $\sum_{k=0}^{+\infty} |c_{k+1} - c_k| < \infty$, *and* $e^k \in H$ *satisfies* (v) $\sum_{k=0}^{+\infty} \|e^k\| < \infty$. *Then the sequence* $\{x^k\}$ *converges strongly to* $P_S u$.

During carefully reading the proof of Theorem X, we discovered that there is a gap (Page 123 line 6 from the bottom). That is, the author used Lemma 2.5 of [1] to yield the following inequality: for an appropriate constant $\gamma > 0$,

$$\begin{aligned} \|x^{k+1} - P_S u\|^2 &\leq \|t_k(u - P_S u) + (1 - t_k)(x^k - P_S u)\|^2 + \gamma\|e^k\| \\ &\leq (1 - t_k)\|x^k - P_S u\|^2 + 2t_k\langle u - P_S u, x^{k+1} - P_S u\rangle + \gamma\|e^k\|. \end{aligned}$$

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But, what we can obtain from Lemma 2.5 of [1] isn't the above, but the following:

$$\begin{aligned}\|x^{k+1} - P_S u\|^2 &= \|J^T_{c_k}(t_k u + (1-t_k)x^k + e^k) - P_S u\|^2 \\ &\le \|t_k(u - P_S u) + (1-t_k)(x^k - P_S u)\|^2 + \gamma\|e^k\| \\ &\le (1-t_k)\|x^k - P_S u\|^2 + 2t_k\langle u - P_S u, t_k(u - P_S u) \\ &\quad + (1-t_k)(x^k - P_S u)\rangle + \gamma\|e^k\|.\end{aligned}$$

In this note, we will correct this gap and remove some unnecessary restriction in Theorem X by means of the property of Hilbert space.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let T be an operator with domain $D(T)$ and range $R(T)$ in H. Recall that T is is said to be *monotone* if its graph $G(T) := \{(x, y) \in H \times H; x \in D(T), y \in Tx\}$ is a monotone set in $H \times H$. That is, T is monotone if and only if $(x, y), (x', y') \in G(T) \Longrightarrow \langle x - x', y - y'\rangle \ge 0$. A monotone operator T is said to be *maximal monotone* if the graph $G(T)$ is not properly contained in the graph of any other monotone operator on H. Let $S = T^{-1}0 = \{x \in D(T); 0 \in Tx\}$. It is known that S is closed and convex. Thus the metric projection P_S from H onto S is well-defined whenever $S \ne \emptyset$. Let $J^T_r = (I + rT)^{-1}$, the resolvent of T. Then the following is well known:

(a) $J^T_r : R(I + rT) \to D(T)$ is nonexpansive (i.e. $\|J^T_r x - J^T_r y\| \le \|x - y\|$ for all $x, y \in R(I + rT)$);
(b) $A^{-1}0 = F(J^T_r) = \{x \in D(J^T_r); J^T_r x = x\}$ for all $r > 0$;
(c) (The Resolvent Identity) For $r > 0$ and $t > 0$ and $x \in H$,

$$J^T_r x = J^T_t\left(\frac{t}{r}x + \left(1 - \frac{t}{r}\right)J^T_r x\right). \tag{1}$$

Lemma 1 [1, Lemma 2.1] *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{k+1} \le (1-\lambda_k)a_k + \lambda_k\beta_k + \sigma_k, \quad \forall k \ge 0,$$

where $\{\lambda_k\}$, $\{\beta_k\}$ and $\{\sigma_k\}$ satisfy the conditions (i) $\sum_{k=0}^{\infty}\lambda_k = \infty$; (ii) *either* $\limsup_{k\to\infty}\beta_k \le 0$ *or* $\sum_{k=0}^{\infty}|\lambda_k\beta_k| < \infty$; (iii) $\sigma_k \ge 0$ *for all k and* $\sum_{k=0}^{\infty}\sigma_k < \infty$. *Then $\{a_k\}$ converges to zero as $k \to \infty$.*

Now we give the revised version of Theorem X.

Theorem 2 *Let T be a maximal monotone operator of a Hilbert space H with $S = T^{-1}0 \ne \emptyset$. Suppose that $\{x^k\}$ is defined by*

$$x^{k+1} = J^T_{c_k}(t_k u + (1-t_k)x^k + e^k).$$

Assumed that $\{t_k\} \subset (0, 1)$ and $\{c_k\} \subset (0, +\infty)$ satisfy the following conditions: (i) $\lim\limits_{k\to\infty} t_k = 0$, (ii) $\sum_{k=0}^{+\infty} t_k = \infty$, (iii) $\sum_{k=0}^{+\infty}|t_{k+1} - t_k| < \infty$, (iv) $0 < \liminf_{k\to\infty} c_k$ *and* $\sum_{k=0}^{\infty}\left|1 - \frac{c_k}{c_{k+1}}\right| < +\infty$ *(or* $\sum_{k=0}^{\infty}|c_k - c_{k+1}| < +\infty$*), and $e^k \in H$ satisfies* (v) $\sum_{k=0}^{+\infty}\|e^k\| < \infty$. *Then the sequence $\{x^k\}$ converges strongly to $P_S u$.*

Proof In the proof of Theorem X (see [1]), the author had already obtained that the $\{x^k\}$ is bounded. Let

$$y^k = t_k u + (1-t_k)x^k + e^k.$$

Then $x^{k+1} = J_{c_k}^T y^k$, and hence both $\{J_{c_k}^T y^k\}$ and $\{y^k\}$ are bounded also.

Since $0 < \liminf_{k\to\infty} c_k$, then there exists $\alpha > 0$ and a positive integer $N > 0$ such that $\forall k > N,\ c_k \geq \alpha$. From the resolvent identity (1), we have

$$J_{c_k}^T y^k = J_{c_{k-1}}^T \left(\frac{c_{k-1}}{c_k} y^k + \left(1 - \frac{c_{k-1}}{c_k}\right) J_{c_k}^T y^k \right).$$

Therefore, for some constant $M > 0$ with $M \geq \max\{\|u\|, \|x^k\|, \|J_{c_k}^T y^k\|, \|y^k\|\}$, we have

$$\begin{aligned}
&\|x^{k+1} - x^k\| = \|J_{c_k}^T y^k - J_{c_{k-1}}^T y^{k-1}\| \\
&\leq \left\| \frac{c_{k-1}}{c_k}(y^k - y^{k-1}) + \left(1 - \frac{c_{k-1}}{c_k}\right)(J_{c_k}^T y^k - y^{k-1}) \right\| \\
&= \left\| \frac{c_{k-1}}{c_k}(y^k - y^{k-1}) + \left(1 - \frac{c_{k-1}}{c_k}\right)(J_{c_k}^T y^k - y^k) + \left(1 - \frac{c_{k-1}}{c_k}\right)(y^k - y^{k-1}) \right\| \\
&\leq \|y^k - y^{k-1}\| + \left|1 - \frac{c_{k-1}}{c_k}\right| \|J_{c_k}^T y^k - y^k\| \\
&\leq |t_k - t_{k-1}|(\|u\| + \|x^{k-1}\|) + (1 - t_k)\|x^k - x^{k-1}\| + \|e^k\| + \|e^{k-1}\| + 2M \left|1 - \frac{c_{k-1}}{c_k}\right| \\
&\leq (1 - t_k)\|x^k - x^{k-1}\| + \|e^k\| + \|e^{k-1}\| + 2M \left(|t_k - t_{k-1}| + \left|1 - \frac{c_{k-1}}{c_k}\right| \right) \\
&\leq (1 - t_k)\|x^k - x^{k-1}\| + \|e^k\| + \|e^{k-1}\| + 2M \left(|t_k - t_{k-1}| + \frac{|c_k - c_{k-1}|}{\alpha} \right).
\end{aligned}$$

It follows from Lemma 1 that

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{2}$$

For each $c \in (0, \alpha)$, we also obtain

$$J_{c_k}^T x^k = J_c^T \left(\frac{c}{c_k} x^k + \left(1 - \frac{c}{c_k}\right) J_{c_k}^T x^k \right).$$

Hence,

$$\begin{aligned}
\|J_{c_k}^T x^k - J_c^T x^k\| &\leq \left\| \frac{c}{c_k} x^k + \left(1 - \frac{c}{c_k}\right) J_{c_k}^T x^k - x^k \right\| \\
&= \left|1 - \frac{c}{c_k}\right| \|J_{c_k}^T x^k - x^k\| \leq \|J_{c_k}^T x^k - x^k\| \\
&\leq \|J_{c_k}^T x^k - x^{k+1}\| + \|x^{k+1} - x^k\|.
\end{aligned}$$

As $\|J_{c_k}^T x^k - x^{k+1}\| \leq t_k\|x^k - u\| + \|e^k\| \to 0$, we have

$$\lim_{k\to\infty} \|J_{c_k}^T x^k - x^{k+1}\| = 0. \tag{3}$$

Thus, it follows that

$$\lim_{k\to\infty} \|J_{c_k}^T x^k - J_c^T x^k\| = 0.$$

Since $\|x^k - J_c^T x^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - J_{c_k}^T x^k\| + \|J_{c_k}^T x^k - J_c^T x^k\|$, then

$$\lim_{k\to\infty} \|x^k - J_c^T x^k\| = 0. \tag{4}$$

Next, we show

$$\limsup_{k\to\infty}\langle u - P_S u, x^k - P_S u\rangle \le 0.$$

By the reflexivity of H and the boundedness of $\{x^k\}$, we can choose a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} \rightharpoonup x^*$ and

$$\limsup_{k\to\infty}\langle u - P_S u, x^k - P_S u\rangle = \lim_{i\to\infty}\langle u - P_S u, x^{k_i} - P_S u\rangle.$$

Then $x^* \in S = T^{-1}0 = F(J_c^T)$. In fact, for some constant $L > 0$, we have

$$\begin{aligned}\|x^{k_i} - x^*\|^2 &= \|x^{k_i} - J_c^T x^*\|^2 + 2\langle x^{k_i} - J_c^T x^*, J_c^T x^* - x^*\rangle + \|x^* - J_c^T x^*\|^2\\ &\le (\|x^{k_i} - J_c^T x^{k_i}\| + \|J_c^T x^{k_i} - J_c^T x^*\|)^2 + 2\langle x^{k_i} - J_c^T x^*, J_c^T x^* - x^*\rangle\\ &\quad + \|x^* - J_c^T x^*\|^2 \le L\|x^{k_i} - J_c^T x^{k_i}\| + \|x^{k_i} - x^*\|^2\\ &\quad + 2\langle x^{k_i} - J_c^T x^*, J_c^T x^* - x^*\rangle + \|x^* - J_c^T x^*\|^2.\end{aligned}$$

Thus,

$$2\langle x^{k_i} - J_c^T x^*, x^* - J_c^T x^*\rangle - \|x^* - J_c^T x^*\|^2 \le L\|x^{k_i} - J_c^T x^{k_i}\|.$$

Let $i \to \infty$ on two sides of the above inequality, we must have $\|x^* - J_c^T x^*\|^2 = 0$. So, $x^* \in S$. Hence, using the well known property of the projection P_S from H to S, we obtain

$$\limsup_{k\to\infty}\langle u - P_S u, x^k - P_S u\rangle = \lim_{i\to\infty}\langle u - P_S u, x^{k_i} - P_S u\rangle = \langle u - P_S u, x^* - P_S u\rangle \le 0.$$

Finally, we have for an appropriate constant $\gamma > 0$,

$$\begin{aligned}\|x^{k+1} - P_S u\|^2 &= \|J_{c_k}^T(t_k u + (1 - t_k)x^k + e^k) - P_S u\|^2\\ &\le \|t_k(u - P_S u) + (1 - t_k)(x^k - P_S u)\|^2 + \gamma\|e^k\|\\ &\le (1 - t_k)^2\|x^k - P_S u\|^2 + t_k^2\|u - P_S u\|^2\\ &\quad + 2t_k(1 - t_k)\langle u - P_S u, x^k - P_S u\rangle + \gamma\|e^k\|.\end{aligned}$$

Therefore,

$$\|x^{k+1} - P_S u\|^2 \le (1 - t_k)\|x^k - P_S u\|^2 + t_k\beta_k + \gamma\|e^k\|, \tag{5}$$

where $\beta_k = t_k\|u - P_S u\|^2 + 2(1 - t_k)\langle u - P_S u, x^k - P_S u\rangle$.

So, an application of Lemma 1 onto (5) yields the desired result. □

Remark It is obvious that Theorem 2 contains Theorem X as a special case. Thus, the gap in the proof of Theorem X is solved.

References

1. Xu, H.K.: A regularization method for the proximal point algorithm. J. Glob. Optim. **36**, 115–125 (2006)